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Large eddy simulation turbulence model with Young measures

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Abstract

We show an alternative proof for the existence of weak solutions to equations describing turbulent flows of fluids. The proof proposed by one of the authors in a previous paper (cf. [A. Świerczewska, Large Eddy Simulation. Existence of Stationary Solutions to a Dynamical Model (submitted for publication). Preprint TU-Darmstadt no. 2314, http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/shadow/pp2314.html]) based on more classical methods. We will use Young measures, which allow us to shorten significantly the limiting procedure in the nonlinear terms and generalize the statement.

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1. Introduction

In the paper [1] the author proved the existence of stationary weak solutions to the equations describing turbulent flows in the three-dimensional torus $\mathbb{T}^3$:

$$\begin{aligned} &v \cdot \nabla v - \operatorname{div} A(y, \nabla^s v) - \nu \Delta v + \nabla q = f, \\ &\operatorname{div} v = 0, \end{aligned} \tag{1.1}$$

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where $v : \mathbb{T}^3 \longrightarrow \mathbb{R}^3$ is the velocity, $q : \mathbb{T}^3 \longrightarrow \mathbb{R}$ is the pressure and $\nabla^s v = \frac{1}{2}(\nabla v + \nabla v^T)$ denotes the symmetric part of the gradient. The operator A is a nonlocal operator, continuous w.r.t. to both variables $y = (\tilde{v}, \nabla^s \tilde{v}, \widetilde{vv}, \widetilde{|\nabla^s v|\nabla^s v})$ and $\nabla^s v$. The tilde denotes a convolution with some smooth function φ, namely $\tilde{u}(x) = \int_{\mathbb{R}^3} u(y)\varphi(x - y)\,\mathrm{d}y$.

The equations come from the Large Eddy Simulation (LES) technique for incompressible flows. The idea of LES has its origin in numerics. Typical for turbulent flows are very different scales, which lead to an increase in the number of numerical operations needed to compute the solution. The LES technique is based on choosing the scales for which the exact solution is computed directly and the scales for which the solution is modelled. The selection of scales is obtained through filtering, i.e. convolving with some function (filter) the Navier–Stokes equations. Adding a constitutive relation representing the contribution of small scales into the flow leads to the Eq. (1.1). The modelling is clearly described in [2–4]. The monograph [4] is a wide description of different modelling techniques in LES. For mathematical results for one of the other approaches we refer to [5,6].

We recall the existence theorem and briefly present the idea of the proof. The set of smooth functions on the torus $\mathbb{T}^3$ can be identified with the set of periodic smooth functions with some period $L \in (0, \infty)$. Therefore $\Omega = (0, L)^3$ is a cube of side L in $\mathbb{R}^3$. By $C^\infty_{\mathrm{per}}(\mathbb{R}^3)$ we denote the set of $C^\infty(\mathbb{R}^3)$ functions, which are periodic with period $L > 0$ in each ith direction, i.e., $u(x + Le_i) = u(x)$, $i = 1, 2, 3$. In the following V denotes the closure of the space

$$\mathcal{V} \equiv \left\{ u \,:\, u \in C^\infty_{\mathrm{per}}(\mathbb{R}^3),\ \operatorname{div} u = 0, \int_\Omega u\,\mathrm{d}x = 0 \right\}$$

with respect to the norm $\|u\|_V = (\int_\Omega |\nabla u|^3\,\mathrm{d}x)^{\frac{1}{3}}$. Its dual space will be denoted by V'.

Theorem 1.1 (*Existence*). *Given $f \in V$ there exists a weak solution to the stationary problem* (1.1), *i.e. the equation*

$$\int_\Omega (v \cdot \nabla \cdot v\phi + A(y, \nabla^s v) \cdot \nabla^s \phi + \nabla v \cdot \nabla \phi - f\phi)\,\mathrm{d}x = 0 \tag{1.2}$$

is satisfied for all $\phi \in V$, where $y = (\tilde{v}, \nabla^s \tilde{v}, \widetilde{vv}, \widetilde{|\nabla^s v|\nabla^s v})$.

Most of the difficulties in the existence proof are concentrated in passing to the limit in the nonlinear term A, given by

$$A(y, \nabla^s v) = c(y)|\nabla^s v|\nabla^s v,$$

with the function c continuous with respect to all variables and satisfying the condition

$$0 < \alpha \le c \le \beta < \infty.$$

With the help of the properties of convolutions, the strong convergence of the subsequence

$$y^{n_k} \to \bar{y} \text{ in } L^\infty(\Omega)$$

is obtained, where $y^n = (\widetilde{v^n}, \nabla^s \widetilde{v^n}, \widetilde{v^n v^n}, \widetilde{|\nabla^s v^n|\nabla^s v^n})$, $\bar{y} = (\tilde{v}, \nabla^s \tilde{v}, \widetilde{vv}, \tilde{\chi})$ and $|\nabla^s v^{n_k}|\nabla^s v^{n_k} \rightharpoonup \chi$ in $L^{\frac{3}{2}}(\Omega)$. First, with the help of the Minty Browder trick, a better characterization of the weak limit $c(y^{n_k})|\nabla^s v^{n_k}|\nabla^s v^{n_k} \rightharpoonup c(\bar{y})|\nabla^s v|\nabla^s v$ in $L^{\frac{3}{2}}(\Omega)$ is achieved.

Noting additionally that the structure of Galerkin approximations yields that

$$\lim_{n\to\infty}\int_\Omega c(\bar{y})|\nabla^s v^n|^3\,\mathrm{d}x = \int_\Omega c(\bar{y})|\nabla^s v|^3\,\mathrm{d}x,$$

allows us to show the convergence of appropriate weighted norms and hence the strong convergence of gradients, which completes the proof.

In the following, we show an alternative way of passing to this limit. It is much shorter; however, it uses more advanced techniques. Below we formulate a more abstract theorem, which solves also the problem of passing to the limit in the turbulent term of system (1.1). Moreover this approach allows one to solve the problem in bounded domains, which is more complicated because of convolving near the boundary. For the sake of conciseness we do not present this case in detail.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set of finite measure and let an operator $A(x,s,\xi) : \Omega\times\mathbb{R}^m\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfy the following conditions:*

(i) *$A(x,s,\xi)$ is a Carathéodory function (measurable w.r.t. x and continuous w.r.t. (s,ξ)).*

(ii) *For all $x\in\Omega$, $s\in\mathbb{R}^m$ and $\xi_1,\xi_2\in\mathbb{R}^n$, $\xi_1\neq\xi_2$*

$$[A(x,s,\xi_1)-A(x,s,\xi_2)]\cdot[\xi_1-\xi_2]>0.$$

(iii) *There exist positive constants c_1, c_2 such that for $p>1$ it holds that*

$$A(x,s,\xi)\cdot\xi \ge c_1|\xi|^p$$

and

$$|A(x,s,\xi)|\le c_2|\xi|^{p-1}.$$

Let $y^n:\Omega\to\mathbb{R}^m$ and $z^n:\Omega\to\mathbb{R}^n$ be sequences of measurable functions such that

(iv) *$y^n\to\bar{y}$ a.e. in Ω,*

(v) *$z^n\rightharpoonup z$ in $L^p(\Omega)$ and $A(x,y^n,z^n)\rightharpoonup\bar{A}$ in $L^{\frac{p}{p-1}}(\Omega)$,*

(vi)

$$\limsup_{n\to\infty}\int_\Omega A(x,y^n,z^n)\cdot z^n\,\mathrm{d}x \le \int_\Omega \bar{A}\cdot z\,\mathrm{d}x.$$

Then there exists a subsequence of (z^n) such that

$$z^n\to z \text{ in } L^p(\Omega).$$

Remark. In the proof of Theorem 1.1 in [1], the Minty Browder trick does not give a good characterization of the term $\tilde{\chi}$ because the operator $A(y,\nabla^s v)$ is not pseudomonotone. However Theorem 1.2 shows that A *is of type M* (i.e. if X is a reflexive Banach space, $A : X\to X'$, $u^n\rightharpoonup u$, $Au^n\rightharpoonup\chi$, $\limsup_{n\to\infty}\langle Au^n,u^n\rangle\le\langle\chi,u\rangle$, then $Au=\chi$). For more details on the theory of pseudomonotone operators we refer the reader to [7]. The idea of Theorem 1.2 is influenced by the compactness properties of Leray–Lions operator (cf. [8, Lemma 5, p. 190]). However their proof follows quite a different path to ours.

2. Young measures tools

For the convenience of the reader we collect below all the necessary tools concerning Young measures used in the proof of Theorem 1.2. For more details and the proofs, we refer the reader to [9, Corollaries 3.2–3.4]; see also [10,11].

Lemma 2.1. *Suppose that the sequence of maps $z^j : \Omega \to \mathbb{R}^d$ generates the Young measure ν. Let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function (i.e. measurable in the first argument and continuous in the second). Let us also assume that the negative part $F^-(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, z^j(x))\,\mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x.$$

If, in addition, the sequence of functions $x \mapsto |F|(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$, then

$$F(\cdot, z^j(\cdot)) \rightharpoonup \int_{\mathbb{R}^d} F(x,\lambda)\,\mathrm{d}\nu_x(\lambda) \text{ in } L^1(\Omega)$$

Remark. The second part of the above theorem can be easily extended to vector valued functions F.

Lemma 2.2. *Let $u^j : \Omega \to \mathbb{R}^d$, $v^j : \Omega \to \mathbb{R}^{d'}$ be measurable and suppose that $u^j \to u$ a.e. while v^j generates the Young measure ν. Then the sequence of pairs $(u^j, v^j) : \Omega \to \mathbb{R}^{d+d'}$ generates the Young measure $x \mapsto \delta_{u(x)} \otimes \nu_x$.*

Lemma 2.3. *Suppose that a sequence z^j of measurable functions from Ω to $\mathbb{R}^d$ generates the Young measure $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$. Then*

$$z^j \to z \text{ in measure if and only if } \nu_x = \delta_{z(x)} \text{ a.e.}$$

3. Proof of Theorem 1.2

We apply Lemma 2.1 to the function $A(x, y^n, z^n) \cdot z^n$. The coercivity condition from assumption (iii) of the theorem assures that the negative part of this function is equal to zero; thus it is certainly weakly relatively compact in $L^1(\Omega)$. This allows us to conclude that

$$\limsup_{n\to\infty} \int_\Omega A(x, y^n, z^n)\cdot z^n\,\mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^n} A(x,s,\xi)\cdot\xi\,\mathrm{d}\mu_x(s,\xi)\,\mathrm{d}x \tag{3.3}$$

where μ_x is the Young measure generated by the sequence (y^n, z^n). However according to Lemma 2.2, we are able to characterize this Young measure more precisely. Since the sequence $y^n \to \bar{y}$ a.e. and z^n generates the Young measure ν_x, the Young measure μ_x generated by this pair satisfies $\mu_x = \delta_{\bar{y}(x)} \otimes \nu_x$. Therefore we can integrate

$$\int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^n} A(x,s,\xi)\cdot\xi\,\mathrm{d}\mu_x(s,\xi)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^n} A(x,\bar{y}(x),\xi)\cdot\xi\,\mathrm{d}\nu_x(\xi)\,\mathrm{d}x. \tag{3.4}$$

In the same way we obtain

$$\int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^n} A(x,s,\xi)\,\mathrm{d}\mu_x(s,\xi)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^n} A(x,\bar{y}(x),\xi)\,\mathrm{d}\nu_x(\xi)\,\mathrm{d}x. \tag{3.5}$$

Since the sequence $|A(x, y^n, z^n)|$ is bounded in $L^{\frac{p}{p-1}}(\Omega)$, it is weakly relatively compact in $L^1(\Omega)$, which implies $\bar{A} = \int_{\mathbb{R}^n} A(x, s, \xi) d\mu_x(s, \xi)$. Thus, from (3.3)–(3.5) and assumption (vi), the following inequality holds:

$$\int_\Omega \int_{\mathbb{R}^n} A(x, \bar{y}(x), \xi)\, \mathrm{d}\nu_x(\xi) \cdot \int_{\mathbb{R}^n} \xi\, \mathrm{d}\nu_x(\xi)\, \mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^n} A(x, \bar{y}(x), \xi) \cdot \xi\, \mathrm{d}\nu_x(\xi)\, \mathrm{d}x. \tag{3.6}$$

From the monotonicity of A w.r.t. the last variable we can deduce that

$$\int_\Omega \int_{\mathbb{R}^n} h(x, \xi)\, \mathrm{d}\nu_x(\xi)\, \mathrm{d}x \geq 0, \tag{3.7}$$

where h is defined by

$$h(x, \xi) := \left[A(x, \bar{y}(x), \xi) - A\left(x, \bar{y}(x), \int_{\mathbb{R}^n} \xi\, \mathrm{d}\nu_x(\xi)\right)\right] \cdot \left[\xi - \int_{\mathbb{R}^n} \xi\, \mathrm{d}\nu_x(\xi)\right].$$

Simple calculations imply that

$$\int_\Omega \int_{\mathbb{R}^n} h(x, \xi)\, \mathrm{d}\nu_x(\xi)\, \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^n} A(x, \bar{y}(x), \xi) \cdot \xi\, \mathrm{d}\nu_x(\xi)\, \mathrm{d}x$$
$$- \int_\Omega \int_{\mathbb{R}^n} A(x, \bar{y}(x), \xi)\, \mathrm{d}\nu_x(\xi) \cdot \int_{\mathbb{R}^n} \xi\, \mathrm{d}\nu_x(\xi)\, \mathrm{d}x,$$

which, together with (3.6), assures that

$$\int_\Omega \int_{\mathbb{R}^n} h(x, \xi)\, \mathrm{d}\nu_x(\xi)\, \mathrm{d}x \leq 0. \tag{3.8}$$

Then, (3.7) and (3.8) imply that $\int_{\mathbb{R}^n} h(x, \xi)\, \mathrm{d}\nu_x(\xi) = 0$ for a.a. $x \in \Omega$. Moreover, since $\nu_x \geq 0$, we have

$$\mathrm{supp}\{\nu_x\} \overset{\text{a.e.}}{=} \left\{\int_{\mathbb{R}^n} \xi\, \mathrm{d}\nu_x(\xi)\right\}.$$

Note that the single point in the right-hand side set is located a.e. in the point $z(x)$, where z is the weak limit of the sequence (z^n). Finally we can conclude that $\nu_x = \delta_{z(x)}$ a.e. A direct application of Lemma 2.3 yields that $z^n \to z$ in measure. Then there exists a subsequence of (z^n) such that $z^n \to z$ a.e. Using the information that $\nu_x = \delta_{z(x)}$ together with Lemma 2.1 and assumption (vi) yields

$$\limsup_{n\to\infty} \int_\Omega A(x, y^n, z^n) \cdot z^n\, \mathrm{d}x \leq \int_\Omega A(x, \bar{y}, z) \cdot z\, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega A(x, y^n, z^n) \cdot z^n\, \mathrm{d}x.$$

Hence we can set $g^n = A(x, y^n, z^n) \cdot z^n$, $g = A(x, \bar{y}, z) \cdot z$ and claim that

$$g^n \geq 0, \qquad g \in L^1(\Omega), \qquad \int_\Omega g^n\, \mathrm{d}x \to \int_\Omega g\, \mathrm{d}x, \qquad g^n \to g \text{ a.e. in } \Omega.$$

Noticing that

$$\int_\Omega |g^n - g|\, \mathrm{d}x = \int_\Omega (g^n - g)\, \mathrm{d}x + 2\int_\Omega \max\{(g - g^n), 0\}\, \mathrm{d}x$$

we conclude by Lebesgue's Dominated Convergence Theorem that $A(x, y^n, z^n)z^n \to A(x, \bar{y}, z)z$ in $L^1(\Omega)$. Thus, by Vitali's Theorem, it is uniformly integrable. Due to the coercivity condition, also the sequence $|z^n|^p$ is uniformly integrable. Using again Vitali's Theorem yields that $z^n \to z$ in $L^p(\Omega)$, which completes the proof. □

4. New proof of Theorem 1.1

We recall the theorem concerning the existence of a solution to the Galerkin approximation:

Theorem 4.1. *For any given $f \in V'$ there exists a solution $\lambda_1^n, \ldots, \lambda_n^n$ (and therefore $v^n \in V^n = \operatorname{span}\{\omega_1, \ldots, \omega_n\}$) to the approximate problem*

$$v^n(x) = \sum_{r=1}^{n} \lambda_r^n \omega_r, \quad \lambda_r^n \in \mathbb{R}, \tag{4.9}$$

$$b(v^n, v^n, \omega_r) + \int_\Omega A(y^n, \nabla^s v^n) \cdot \nabla^s \omega_r \, dx + \nu(\nabla v^n, \nabla \omega_r) = \langle f, \omega_r \rangle \tag{4.10}$$

for $r = 1, \ldots, n$, where we denote $b(u, v, w) = \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, dx$.

From the energy estimate we conclude the convergence (even if the convergence holds only for a subsequence, throughout this proof we will not use subsequence indexes)

$$v^n \rightharpoonup v \text{ in } V,$$

which implies

$$v^n \to v \text{ in } L^q(\Omega) \qquad \text{for } 1 \le q < \infty,$$
$$b(v^n, v^n, \phi) \to b(v, v, \phi) \qquad \text{for all } \phi \in V$$

and we can conclude the existence of $\bar{A} \in L^{\frac{3}{2}}(\Omega)$, $\bar{y} \in L^\infty(\Omega)$ such that

$$A(y^n, \nabla^s v^n) \rightharpoonup \bar{A} \text{ in } L^{\frac{3}{2}}(\Omega), \qquad y^n \to \bar{y} \text{ in } L^\infty(\Omega)$$

and we recall the notation $y^n = (\widetilde{v^n}, \nabla^s \widetilde{v^n}, \widetilde{v^n v^n}, |\nabla^s \widetilde{v^n | \nabla^s} v^n)$, $\bar{y} = (\tilde{v}, \nabla^s \tilde{v}, \widetilde{vv}, \tilde{\chi})$. To achieve the last convergence we use the regularizing properties of convolutions. Hence we apply now Theorem 1.2 to the operator

$$A(y, \nabla^s v) : (\mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3) \times \mathbb{S}^3 \to \mathbb{S}^3,$$

where by $\mathbb{S}^3$ we denote the set of symmetric matrices in $\mathbb{R}^{3\times 3}$. We identify $\mathbb{S}^3$ with $\mathbb{R}^6$ and $(\mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3) \times \mathbb{S}^3$ with $\mathbb{R}^{21} \times \mathbb{R}^6$. In the notation of Theorem 1.2 y^n is as denoted above and converges to $\bar{y}$ a.e.; $z^n = \nabla^s v^n$. The operator A does not depend on x and is continuous w.r.t. all other variables; the assumptions (ii)–(iii) are certainly satisfied. The assumption (vi) holds while the approximate equation yields

$$\int_\Omega A(y^n, z^n) \cdot z^n \, dx = -\nu \|z^n\|_{L^2}^2 + \int_\Omega f v^n \, dx,$$

and thus due to the lower semicontinuity of the norm it holds that

$$\limsup_{n\to\infty} \int_\Omega A(y^n, z^n) \cdot z^n \, dx \le -\nu \|z\|_{L^2}^2 + \int_\Omega f v \, dx = \int_\Omega \bar{A} \cdot z \, dx.$$

Thus according to Theorem 1.2 we obtain that $z^n \to z$ a.e. in Ω and strongly in $L^3(\Omega)$. This improves the information on the convergence of y^n; namely we can claim that $|\nabla^s \widetilde{v^n | \nabla^s} v^n \to |\nabla^s \widetilde{v | \nabla^s} v$ in $L^\infty(\Omega)$ and thus $\bar{y} = y$. Finally this yields that $A(y^n, z^n) \to A(y, z)$ strongly in $L^{\frac{3}{2}}(\Omega)$, which concludes the proof. $\square$

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